MATH 1010E Week 4 (Martin Li)

Last time ...

- f differentiable at $c \Rightarrow f$ cts at c
- $(f \pm g)' = f' \pm g'$; $(kf)' = kf'$ k = constant.

Derivatives of Piecewise-defined Functions

E.g. Find the derivative $f'(x)$ for the function

$$f(x) := \begin{cases} 5-2x & \text{when } x<0 \\ x^2-2x+5 & \text{when } x \geq 0 \end{cases}$$

Sol: Case 1: $x<0$

$$f'(x) = (5-2x)' = -2.$$

Case 2: $x>0$

$$f'(x) = (x^2-2x+5)' = 2x-2$$

Case 3: $x=0$. (use def^n). $f(0) = 0^2 - 2\cdot 0 + 5 = 5.$

$$f'(0) = \lim_{h \to 0} \frac{f(h)-f(0)}{h} = -2$$

left hand: $\displaystyle \lim_{h \to 0^-} \frac{(\cancel{5}-2\cancel{h})-\cancel{5}}{\cancel{h}} = -2$

$\|$

right hand: $\displaystyle \lim_{h \to 0^+} \frac{(h^{\cancel{2}}-2\cancel{h}+\cancel{5})-\cancel{5}}{\cancel{h}} = -2$

$$\Rightarrow f'(x) = \begin{cases} -2 & \text{when } x \leq 0 \\ 2x-2 & \text{when } x>0 \end{cases}$$

Remark: f' is cts at 0.
(not always true).

E.g. 2 : let

$$f(x) = \begin{cases} ax+b & \text{when } x \leq -1 \\ ax^3 + x + 2b & \text{when } x > -1. \end{cases}$$

Find $a, b \in \mathbb{R}$ s.t. f is differentiable for all $x \in \mathbb{R}$.

Sol: ~~obverse:~~ Observation : f is diff. for $x \neq -1$

① If f is cts at $x = -1$, then

$$\lim_{x \to -1} f(x) = f(-1)$$

Now, $f(-1) = -a + b$.

$$\lim_{x \to -1^+} f(x) = a(-1)^3 + (-1) + 2b = -a + 2b - 1$$

$$\lim_{x \to -1^-} f(x) = a(-1) + b = -a + b.$$

cts $\iff$ $\boxed{-a + b = -a + 2b - 1}$ $\Rightarrow$ $\boxed{b = 1}$

② If f is diff. at $x = -1$, then

$$\lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \quad \text{exists.}$$

$\iff$ $\lim\limits_{h \to 0^+}$ and $\lim\limits_{h \to 0^-}$ exists & equal.

left-hand: let $x = -1+h$, $f(-1) = -a+1$

$$\lim_{h\to 0^-} \frac{f(\overbrace{-1+h}^{x}) - f(-1)}{\underbrace{h}_{x-(-1)}} = \lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} .$$

$$= \lim_{x\to -1^-} \frac{(ax+\cancel{1}) - (-a+\cancel{1})}{x+1} = a$$

right-hand:

$$\lim_{h\to 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{x\to -1^+} \frac{f(x)-f(-1)}{x+1}$$

$$= \lim_{x\to -1^+} \frac{(ax^3+x+2)-(-a+1)}{x+1}$$

$$= \lim_{x\to -1^+} \frac{ax^3+x+(1+a)}{x+1}$$

$x^3+1 = (x+1)(x^2-x+1)$

$$= \lim_{x\to -1^+} \frac{a(\overbrace{\cancel{x^3+1}}^{x^2-x+1}) + \overbrace{\cancel{(x+1)}}^{1}}{\cancel{x+1}}$$

$$= a(1+1+1)+1$$

$$= 3a+1 .$$

lim exists $\iff \boxed{a = 3a+1} \implies \boxed{a = -\frac{1}{2}}$

Differentiation Rules II

(1) Product rule

$$\big[f(x)\,g(x)\big]' = f'(x)\,g(x) + f(x)\,g'(x)$$

(2) Quotient rule

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Remark: $[k f(x)]' = k f'(x)$ when k = constant

follows from product rule by taking $g(x) = k$ $\;(g'(x) = 0)$.

Examples

(i) $$\frac{d}{dx}\Big[(x+2)(x^2+1)\Big] = (x+2)'(x^2+1) + (x+2)(x^2+1)'$$

$$= (x^2+1) + (x+2)(2x)$$

$$= 3x^2 + 4x + 1 \quad \#$$

(ii) $$\frac{d}{dx}\Big(\underbrace{\sin x \cos x}_{\frac{1}{2}\sin 2x}\Big) = (\sin x)'(\cos x) + (\sin x)(\cos x)'$$

$$= \cos x \cdot \cos x + \sin x\,(-\sin x)$$

$$= \cos^2 x - \sin^2 x = \cos 2x. \quad \#$$

(iii) $$\frac{d}{dx}\left(\frac{x^2+1}{x+1}\right) = \frac{(x+1)(x^2+1)' - (x+1)'(x^2+1)}{(x+1)^2}$$

$$= \frac{(x+1)(2x) - (x^2+1)}{(x+1)^2} \qquad \left(\begin{array}{c}\text{for}\\ x \neq -1\end{array}\right)$$

$$= \frac{x^2+2x-1}{(x+1)^2} \quad \#$$

(iv) $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = \dfrac{x\cos x - \sin x}{x^2}$ #.

(for $x \neq 0$)

Q: $\lim\limits_{x\to 0} \dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right)$ exists ?. Related to $f'(0)$?

cannot: $\dfrac{d}{dx}\left(\lim\limits_{x\to 0}\dfrac{\sin x}{x}\right)$ X.

define $f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$

Q: $\dfrac{d}{dx}(\tan x) = ?$; $\dfrac{d}{dx}(\sec x) = ?$.

$\dfrac{d}{dx}(\cosh x) = ?$; $\dfrac{d}{dx}(\coth x) = ?$.

Proof of Product Rule: $(fg)' = f'g + fg'$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$[f(x)g(x)]' = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

at x

$$= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\underbrace{g(x+h)}_{g(x)}\cdot\underbrace{\frac{f(x+h)-f(x)}{h}}_{f'(x)}\right] + \underbrace{\lim_{h\to 0}\left[f(x)\cdot\frac{g(x+h)-g(x)}{h}\right]}_{f(x)\,g'(x)}$$

g diff. at x

$\Downarrow$

g cts at x

true

Proof of Quotient Rule $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ when $g \neq 0$.

$$\left(\frac{f}{g}\right)' := \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x+h)\, g(x)}$$

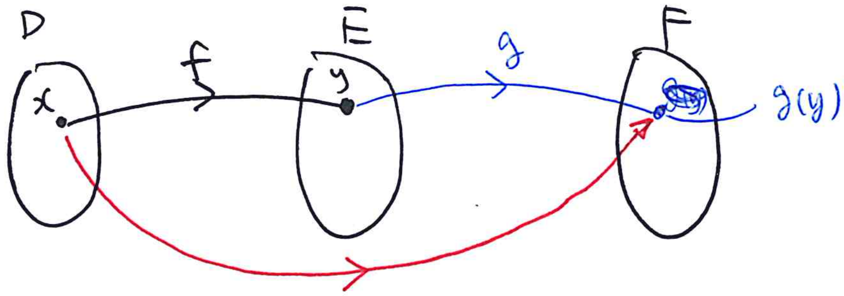
$$= \lim_{h \to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x)}{h} + \frac{f(x)g(x) - f(x)g(x+h)}{h}}{g(x+h)\, g(x)}$$

$$= \lim_{h \to 0} \frac{g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h}}{g(x+h)\, g(x)}$$

$$= \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$$ #.

Composite Functions

Given a function $f: D \to E$, $g: E \to F$



$g \circ f : D \to F$

$g \circ f(x) := g(f(x))$ composite function.

E.g. $f(x) := \cos x$ $\quad f: \mathbb{R} \to \mathbb{R}$

$g(y) := y^2$ $\quad g: \mathbb{R} \to \mathbb{R}$

$g \circ f(x) = g(f(x)) = g(\cos x) = \cos^2 x$.

$\neq$

$f \circ g(y) = f(g(y)) = f(y^2) = \cos(y^2)$

In general $\boxed{g \circ f \neq f \circ g}$

Thm: If f is cts at x_0

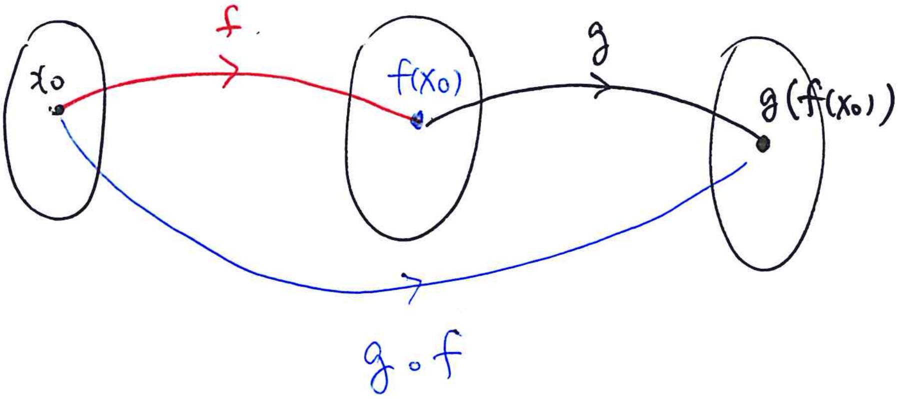
and g is cts at $f(x_0)$

then $g \circ f$ is cts at x_0

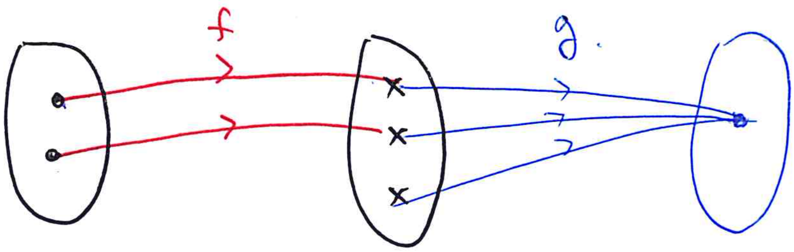
E.g. $f(x) = \cos x^2$

is cts at $x = 0$.

$\Rightarrow \lim_{x \to 0} \cos x^2 = \cos 0^2 = 1$.



$g \circ f$



Differentiation Rule III – "Chain Rule"

Thm: If f diff. at c

g diff. at $f(c)$

then $g \circ f$ is diff. at C

$$\boxed{(g \circ f)'(c) = g'(f(c)) \cdot f'(c)}$$

$$\boxed{\text{"}\ \frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx}\ \text{"}}$$

E.g. (1) $\dfrac{d}{dx}\left(\cos x^2\right)$

Let $f(x) = x^2$, $g(y) = \cos y$ $\Rightarrow$ $g \circ f(x) = \cos x^2$

Now, $f'(x) = 2x$, $g'(y) = -\sin y$. $\Rightarrow (g \circ f)'(x)$

$$= g'(f(x)) \cdot f'(x)$$

$$= (-\sin x^2) \cdot (2x)$$

$$= -2x \sin x^2 \ \#$$

(2) $\dfrac{d}{dx}(x^2+1)^7 = 7(x^2+1)^6 \cdot 2x$ #

$y = x^2+7$: $\dfrac{d}{dx}(x^2+1)^7 = \dfrac{d}{dx}y^7 = \left(\dfrac{d}{dy}y^7\right)\left(\dfrac{dy}{dx}\right) = 7y^6 \cdot 2x = 7(x^2+7)^6 \cdot 2x$

(3) $\dfrac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x} \cdot \cos x$ #.

(4) $\dfrac{d}{dx}\left(e^{\sin(x^2)}\right) = e^{\sin x^2} \cdot \cos x^2 \cdot 2x$ #.

Q: $\dfrac{d}{dx}\left(\sqrt{x+\sqrt{x}}\right) = ?$

$\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = ?$

Recall: (Chain Rule)

$$\boxed{\frac{d}{dx}\, g(f(x)) = g'(f(x)) \cdot f'(x)}$$

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx}$$

Proof:

$$\left.\frac{d}{dx}\right|_{x=x_0} g(f(x)) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \right]$$

Take $y = f(x)$

$y_0 = f(x_0)$

$$= \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= g'(y_0) \cdot f'(x_0)$$

$$= g'(f(x_0)) \cdot f'(x_0)$$

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(Since f is cts,

so $\lim_{x \to x_0} f(x) = f(x_0)$

ie as $x \to x_0$, $y \to y_0$)

Derivative of an inverse

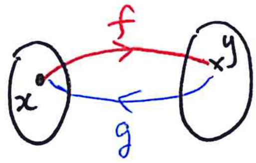
Q: If f has an inverse f^{-1}, then what is $(f^{-1})'$?

Thm: Let $f: (a,b) \longrightarrow (c,d)$, 1-1, onto, differentiable,

then the inverse $f^{-1} = g: (c,d) \to (a,b)$ is differentiable

[Q: $f' \neq 0$ if f is 1-1, onto?] and

$$\boxed{g'(y) = \frac{1}{f'(g(y))}}$$



Pf: By def^{n} of inverse, $f(g(y)) = y$ for all y and $g(f(x)) = x$ for all x

Diff. on both sides. w.r.t y,

$$f'(g(y))\, g'(y) = 1 \quad \Rightarrow \quad g'(y) = \frac{1}{f'(g(y))}.$$

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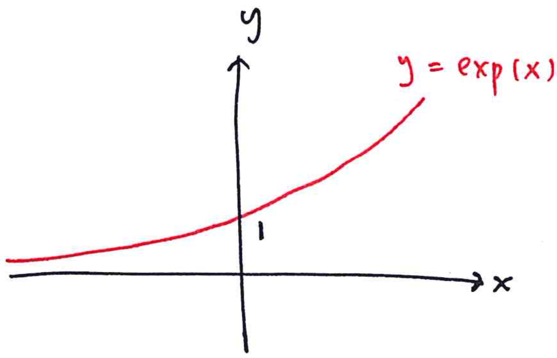
Example 1: $\exp(x) : \mathbb{R} \longrightarrow (0,\infty)$ 1-1, onto

has inverse $\ln(y) : (0,\infty) \to \mathbb{R}$



$$\boxed{\frac{d}{dy}\ln(y) = \frac{1}{y}}$$

Know: $\frac{d}{dx}\exp(x) = \exp(x)$

$y = \exp(x)$

$$\underbrace{\frac{d}{dy}\ln(y)}_{\text{function of } y} = \underbrace{\frac{1}{\exp(x)}}_{\text{function of } x} = \frac{1}{y}.$$

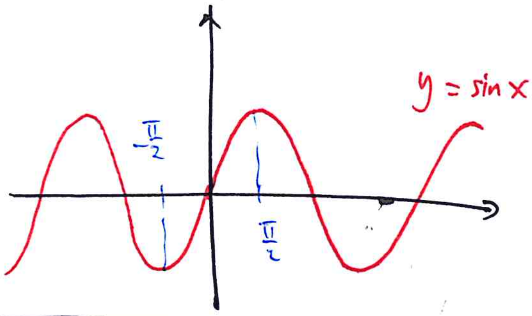
\#

Example 2: $\frac{d}{dx}(\sin^{-1}x) = ? = \frac{d}{dy}(\sin^{-1}y)$

$\sin(x) : \mathbb{R} \longrightarrow [-1,1]$ not 1-1

restrict to smaller domain $\Rightarrow$ 1-1, onto

$\sin(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow (-1,1)$



inverse $\Rightarrow$ $\sin^{-1}(y) : (-1,1) \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ $\boxed{y = \sin x}$

$$\frac{d}{dy}\sin^{-1}(y) = \frac{1}{\frac{d}{dx}(\sin x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.$$

\#

Q: what happens at $y = \pm 1$?

A delicate example

Consider

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

Q: (i) f is differentiable for every $x \in \mathbb{R}$. (including $x=0$)

(ii) $f'(x) : \mathbb{R} \to \mathbb{R}$

$\lim_{x \to 0} f'(x) = f'(0)$? Is f' cts at $x=0$?

(iii) Sketch the graph of f.

Q: $(\tan^{-1}x)'$, $(\cos^{-1}x)'$, $(\cosh^{-1}x)'$ ....

More examples

(1) $\frac{d}{dx}\left(\ln\left(x+\sqrt{1+x^2}\right)\right)$ for $x \in \mathbb{R}$.

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \underbrace{\frac{d}{dx}\left(x+\sqrt{1+x^2}\right)}_{1+\underbrace{\frac{d}{dx}\sqrt{1+x^2}}_{\frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}}\cdot\cancel{2}x}}$$

$\frac{d}{dx}x^n = n x^{n-1}$

$n \in \mathbb{R}$.

$$= \frac{1}{\cancel{x+\sqrt{1+x^2}}}\left[\underbrace{1+x(1+x^2)^{-\frac{1}{2}}}_{\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}}\right] = \frac{1}{\sqrt{1+x^2}} \quad \#.$$

(2). $\frac{d}{dx}(3^x) = \frac{d}{dx}\left(\exp(x\ln 3)\right)$

$\boxed{a^x := \exp(x\ln a)}$

$a \in \mathbb{R}$, $x \in \mathbb{R}$

$$= \exp(x\ln 3)\cdot\ln 3$$

$$= (\ln 3)\cdot 3^x \quad \#$$

(3) $\frac{d}{dx}(x^x) = \frac{d}{dx}\left(\exp(x\ln x)\right)$ $\boxed{x>0}$

$$= \exp(x\ln x)\left[1\cdot\ln x + x\cdot\frac{1}{x}\right]$$

$$= (1+\ln x)\,x^x \quad \#.$$

(4) $\frac{d}{dx}\left(x^{x^x}\right) = ?$